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Journal of Geometry and Physics 56 (2006) 2403–2414
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Geometric structures of vectorial type

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Received 5 October 2005; accepted 29 December 2005
Available online 8 February 2006

Abstract

We study geometric structures of $\mathcal{W}_4$-type in the sense of A. Gray on a Riemannian manifold. If the structure group $G \subset SO(n)$ preserves a spinor or a non-degenerate differential form, its intrinsic torsion Γ is a closed 1-form (Proposition 2.1 and Theorem 2.1). Using a G-invariant spinor we prove a splitting theorem (Proposition 2.2). The latter result generalizes and unifies a recent result obtained in [S. Ivanov, M. Parton, P. Piccinni, Locally conformal parallel G_2- and Spin(7)-structures, mathdg/0509038], where this splitting has been proved in dimensions $n = 7, 8$ only. Finally we investigate geometric structures of vectorial type and admitting a characteristic connection ∇^c. An interesting class of geometric structures generalizing Hopf structures are those with a ∇^c-parallel intrinsic torsion Γ. In this case, Γ induces a Killing vector field (Proposition 4.1) and for some special structure groups it is even parallel.

MSC: primary 53 C 25; secondary 81 T 30

Keywords: Connection with torsion; Intrinsic torsion; Parallel spinor; Special geometric structure; Hopf structure; Weyl geometry

1. Adapted connections of a geometric structure of vectorial type

Fix a subgroup $G \subset SO(n)$ of the special orthogonal group and decompose the Lie algebra $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ into the Lie algebra $\mathfrak{g}$ of G and its orthogonal complement $\mathfrak{m}$. The different geometric types of G-structures on a Riemannian manifold correspond to the irreducible G-components of the representation $\mathbb{R}^n \otimes \mathfrak{m}$. Indeed, consider an oriented Riemannian manifold

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2005.12.007

(M^n, g) and denote its Riemannian frame bundle by $\mathcal{F}(M^n)$. It is a principal SO(n)-bundle over M^n. A G-structure is a reduction $\mathcal{R} \subset \mathcal{F}(M^n)$ of the frame bundle to the subgroup G. The Levi-Civita connection is a 1-form Z on $\mathcal{F}(M^n)$ with values in the Lie algebra $\mathfrak{so}(n)$. We restrict the Levi-Civita connection to $\mathcal{R}$ and decompose it with respect to the decomposition of the Lie algebra $\mathfrak{so}(n)$,

$$Z|_{T(\mathcal{R})} := Z^* \oplus \Gamma.$$

Then, Z^* is a connection in the principal G-bundle $\mathcal{R}$ and Γ is a 1-form on M^n with values in the associated bundle $\mathcal{R} \times_{\mathrm{G}} \mathfrak{m}$. This 1-form, or more precisely the G-components of the element $\Gamma \in \mathbb{R}^n \otimes \mathfrak{m}$, characterizes the different types of non-integrable G-structures (see [12]). The 1-form Γ is called the *intrinsic torsion* of the G-structure. There is a second notion for G-structures, namely that of *characteristic connection* and its *characteristic torsion*. By definition, a characteristic connection is a G-connection ∇^c with totally skew-symmetric torsion tensor. Typically, not every type of G-structure admits a characteristic connection. In order to formulate the condition, we embed the space of all 3-forms into $\mathbb{R}^n \otimes \mathfrak{m}$ using the morphism

$$\Theta : \Lambda^3(\mathbb{R}^n) \longrightarrow \mathbb{R}^n \otimes \mathfrak{m}, \quad \Theta(\mathrm{T}) := \sum_{i=1}^{n} e_i \otimes \mathrm{pr}_{\mathfrak{m}}(e_i \lrcorner \mathrm{T}).$$

A G-structure admits a characteristic connection ∇^c if and only if the intrinsic torsion Γ belongs to the image of the Θ. In this case, the characteristic torsion is the pre-image of the intrinsic torsion (see [12])

$$2\Gamma = -\Theta(\mathrm{T}^c) \quad \text{and} \quad \nabla^c_X Y = \nabla^g_X Y + \frac{1}{2}\mathrm{T}^c(X, Y, -).$$

For different geometric structures, the characteristic torsion form has been computed explicitly in terms of the underlying geometric data. Formulas of that type are known for almost hermitian structures, almost contact metric structures and G_2- and Spin(7)-structures in dimensions seven and eight. In case of a Riemannian naturally reductive space $M^n = \mathrm{G}_1/\mathrm{G}$, we obtain a G-reduction $\mathcal{R} := \mathrm{G}_1 \subset \mathcal{F}(M^n)$ of the frame bundle. Then the characteristic connection of the G-structure coincides with the *canonical* connection of the reductive space. In this sense, we can understand the characteristic connection of a Riemannian G-structure as a generalization of the canonical connection of a Riemannian naturally reductive space. The canonical connection of a naturally reductive space has parallel torsion form and parallel curvature tensor, $\nabla^c \mathrm{T}^c = 0 = \nabla^c \mathrm{R}^c$. For arbitrary G-structures and their characteristic connections, these properties do not hold any longer. Corresponding examples are discussed in [13]. The space $\mathbb{R}^n \otimes \mathfrak{m}$ contains $\mathbb{R}^n$ in a natural way,

$$\Theta_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^n \otimes \mathfrak{m}, \quad \Theta_1(\Gamma) = \sum_{i=1}^{n} e_i \otimes \mathrm{pr}_{\mathfrak{m}}(e_i \wedge \Gamma).$$

The class of geometric structures that we will study in this paper is the following one.

Definition 1.1. Let M^n be an oriented Riemannian manifold and denote by $\mathcal{F}(M^n)$ its frame bundle. A geometric structure $\mathcal{R} \subset \mathcal{F}(M^n)$ is called of *vectorial type* if its intrinsic torsion belongs to $\Gamma \in \mathbb{R}^n \subset \mathbb{R}^n \otimes \mathfrak{m}$.

Remark 1.1. These geometric structures are usually called $\mathcal{W}_4$-structures. They occur in the description of almost hermitian manifolds, of G_2-structures in dimension seven, of Spin(7)-structures in dimension eight and Spin(9)-structures in dimension sixteen (see [10]).

We will identify vectors field on the Riemannian manifold (M^n, g) with 1-forms. Denoting the vector field corresponding to the intrinsic torsion by Γ, too, we obtain the following formula for the intrinsic torsion defined by a vector field Γ:

$$\Gamma(X) = \mathrm{pr}_{\mathfrak{m}}(X \wedge \Gamma).$$

Of course, a geometric structure of vectorial type does not have to admit a characteristic connection. It depends on the decomposition of the G-representation $\Lambda^3(\mathbb{R}^n)$. For example, consider the subgroup $\mathrm{SO}(3) \subset \mathrm{SO}(5)$ defined by the five-dimensional, real representation of SO(3). Then $\Lambda^3(\mathbb{R}^5)$ splits into a three-dimensional and a seven-dimensional SO(3)-representation, i.e., $\Theta(\Lambda^3(\mathbb{R}^5)) \subset \mathbb{R}^5 \otimes \mathfrak{m}$ and $\mathbb{R}^5 \subset \mathbb{R}^5 \otimes \mathfrak{m}$ are complementary subspaces. A similar situation occurs for the subgroups $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ and for $F_4 \subset \mathrm{SO}(26)$ (see [12]). On the other hand, many interesting geometric structures of vectorial type admit characteristic connections. This situation occurs for example for the subgroups $\mathrm{U}(n) \subset \mathrm{SO}(2n)$, $G_2 \subset \mathrm{SO}(7)$ and $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. The corresponding characteristic torsion has been computed explicitly for these cases in [13].

The first observation is a link to E. Cartan (see [1,8]), who classified the types of metric connections. There are two special classes. The first class is that of metric connections of vectorial type; the second class is that of metric connections with a totally skew-symmetric torsion. There also exists a third class, but it does not have a direct geometric interpretation. The geodesic flow of metric connections of vectorial type has been investigated in [8] and [2]. On the other hand, the geodesic flow of metric connections with totally skew-symmetric torsion coincides with the Riemannian geodesic flow.

Proposition 1.1. *If a* G*-structure is of vectorial type, then there exists a unique metric connection* ∇^{vec} *of vectorial type in the sense of Cartan and preserving the* G*-structure. The formula is*

$$\nabla^{\mathrm{vec}}_X Y = \nabla^g_X Y - g(X,Y)\cdot\Gamma + g(Y,\Gamma)\cdot X.$$

Conversely, if a G*-structure* $\mathcal{R}$ *admits a connection of vectorial type in the sense of Cartan, then* $\mathcal{R}$ *is of vectorial type in our sense.*

Proof. The Levi-Civita connection splits into

$$Z(X) = Z^*(X) + \mathrm{pr}_{\mathfrak{m}}(X \wedge \Gamma) = Z^*(X) - \mathrm{pr}_{\mathfrak{g}}(X \wedge \Gamma) + X \wedge \Gamma.$$

The formula $\beta(X) := \mathrm{pr}_{\mathfrak{g}}(X \wedge \Gamma)$ defines a 1-form with values in the Lie algebra $\mathfrak{g}$. Therefore, the connection

$$Z^{\mathrm{vec}}(X) := Z(X) - X \wedge \Gamma = Z^*(X) - \mathrm{pr}_{\mathfrak{g}}(X \wedge \Gamma)$$

is a G-connection. It is of vectorial type in the sense of Cartan. Suppose vice versa that there exists a G-connection Z^{**} of vectorial type. We compare it with the Levi-Civita connection and obtain the relation

$$Z^{**}(X) = Z(X) + X \wedge \Gamma.$$

Moreover, the definition of the 1-form Γ as well as the G-connection Z^* yields the equation

$$Z(X) = Z^*(X) + \Gamma(X).$$

Finally, since Z^{**} preserves the G-structure, there exists a 1-form β with values in the Lie algebra $\mathfrak{g}$ such that

$$Z^{**}(X) = Z^{*}(X) + \beta(X).$$

Combining these three formulas we obtain, for any vector X, the equation

$$\beta(X) = \Gamma(X) + X \wedge \Gamma.$$

We now take the projection onto $\mathfrak{m}$. Since $\beta(X)$ belongs to the Lie algebra $\mathfrak{g}$, we conclude that Γ should be in the image of $\mathbb{R}^n \subset \mathbb{R}^n \otimes \mathfrak{m}$. □

Let Γ be a vector field on a Riemannian manifold (M^n, g). Then we define a metric connection ∇^{vec} as before. Its holonomy group is a subgroup of $\text{SO}(n)$ and its holonomy bundle is a reduction of the frame bundle $\mathcal{F}(M^n)$. Obviously, the corresponding structure is of vectorial type. Therefore, any vector field can occur. However, if the structure group G is fixed, then we obtain restrictions for the possible vector field Γ. In the next section we will explain the corresponding results.

A geometric structure of vectorial type induces a triple (M^n, g, Γ) consisting of a Riemannian manifold and a vector field. A similar situation occurs in Weyl geometry (see [7,14]). A Weyl structure is a pair consisting of a conformal class of metric and a torsion free connection preserving the conformal structure. Choosing a metric g in the conformal class, the connection defines a vector field and the corresponding covariant derivative on vectors is defined by the formula

$$\nabla^{\text{w}}_X Y = \nabla^g_X Y + g(X, \Gamma) \cdot Y + g(Y, \Gamma) \cdot X - g(X, Y) \cdot \Gamma.$$

Weyl geometry deals with the geometric properties of these connections. The two connections ∇^{vec} and ∇^{w} are different. The Weyl connection does not preserve any Riemannian geometric structure; moreover, it is torsion free. However, the curvature tensors and the Ricci tensors are closely related:

$$\mathcal{R}^{\text{vec}}(X, Y)Z = \mathcal{R}^{\text{w}}(X, Y)Z - \mathrm{d}\Gamma(X, Y) \cdot Z, \quad \text{Ric}^{\text{vec}} = \text{Ric}^{\text{w}} + \mathrm{d}\Gamma.$$

In particular, the symmetric parts of the Ricci tensors coincide. If one can prove that certain geometric structures induce Weyl–Einstein structures, one can apply several results known in Weyl geometry. Examples of this approach can be found in Theorem 2.1 and Proposition 2.2. Otherwise, the topics are quite different.

Of course, a conformal change of G-structures is again possible. Let us discuss it. The total space $\mathcal{R} \subset \mathcal{F}(M^n, g)$ of a G-structure consists of n-tuples $(e_1, e_2, \ldots, e_n)$ of orthonormal vectors tangent to M^n. Let $g^* := e^{2f} g$ be a conformal change of the metric. Then we define a new G-structure $\mathcal{R}^* \subset \mathcal{F}(M^n, g^*)$ by

$$\mathcal{R}^* = \{(e^{-f} \cdot e_1, e^{-f} \cdot e_2 \ldots, e^{-f} \cdot e_n) : (e_1, e_2, \ldots, e_n) \in \mathcal{R}\}.$$

The intrinsic torsion changes by the element $\mathrm{d}f \in \mathbb{R}^n \subset \mathbb{R}^n \otimes \mathfrak{m}$, $\Gamma^* = \Gamma + \mathrm{d}f$. In particular, the conformal change of a geometric structure of vectorial type is again of vectorial type. Moreover, the differentials $\mathrm{d}\Gamma = \mathrm{d}\Gamma^*$ coincide. On the other hand, starting with an arbitrary geometric structure on a compact manifold, the equation

$$0 = \delta^{g^*}(\Gamma^*) = \delta^g(\Gamma) + \Delta(f) + (n-2) \cdot ((\mathrm{d}f, \Gamma) + \|\mathrm{d}f\|^2)$$

has a unique solution $f = -\Delta^{-1}(\delta^g(\Gamma))$. Consequently, an arbitrary geometric structure of vectorial type on a compact manifold admits a conformal change such that the new 1-form is coclosed (see [14,7]). In principle, one can reduce the investigation of geometric structures of vectorial type on compact manifolds to those structures with a coclosed form, $\delta^g(\Gamma) = 0$.

2. Parallel forms and spinors

Let $\Omega^k \in \Lambda^k(\mathbb{R}^n)$ be a G-invariant k-form. It defines a k-form on any Riemannian manifold with a fixed G-structure which is parallel with respect to any G-connection. The Lie algebra $\mathfrak{so}(n) = \Lambda^2(\mathbb{R}^n)$ acts on the vector space $\Lambda^k(\mathbb{R}^n)$ via the formula

$$\rho_*(\omega^2)\Omega^k = \sum_{i=1}^{n}(e_i \lrcorner\, \omega^2) \wedge (e_i \lrcorner\, \Omega^k), \quad \omega^2 \in \mathfrak{so}(n).$$

Consequently, we can compute the Riemannian covariant derivative of Ω^k,

$$\begin{aligned}\nabla^g_X \Omega^k &= \rho_*(\mathrm{pr}_{\mathfrak{m}}(X \wedge \Gamma))\Omega^k = \rho_*(X \wedge \Gamma)\Omega^k = \sum_{i=1}^{n}(e_i \lrcorner\, (X \wedge \Gamma)) \wedge (e_i \lrcorner\, \Omega^k)\\ &= \Gamma \wedge (X \lrcorner\, \Omega^k) - X \wedge (\Gamma \lrcorner\, \Omega^k).\end{aligned}$$

The differential of Ω^k as well as its codifferential are given by

$$\begin{aligned}\mathrm{d}\Omega^k &= \sum_{i=1}^{n} e_i \wedge \nabla^g_{e_i}\Omega^k = -k \cdot (\Gamma \wedge \Omega^k),\\ \delta^g \Omega^k &= -\sum_{i=1}^{n} e_i \lrcorner\, \nabla^g_{e_i}\Omega^k = (n-k) \cdot (\Gamma \lrcorner\, \Omega^k).\end{aligned}$$

In particular, for any geometric structure of vectorial type and any G-invariant form Ω^k we have

$$\nabla^g_\Gamma \Omega^k = 0 \quad \text{and} \quad \mathrm{d}\Gamma \wedge \Omega^k = 0.$$

From these equations we see that Γ is automatically closed if the k-form Ω^k – treated as a linear map defined on 2-forms – has trivial kernel.

Proposition 2.1. *Let* G $\subset$ SO(n) *be a subgroup such that*

(1) *there exists a* G*-invariant differential form* Ω^k *of some degree k, and*
(2) *the multiplication* $\Omega^k : \Lambda^2(\mathbb{R}^n) \to \Lambda^{k+2}(\mathbb{R}^n)$ *is injective.*

Then, for any G*-structure of vectorial type, the 1-form* Γ *is closed;* $\mathrm{d}\Gamma = 0$.

Remark 2.1. The groups $G_2 \subset SO(7)$ and $\mathrm{Spin}(7) \subset SO(8)$ satisfy the conditions of the Proposition. Consequently, it generalizes results of Cabrera (see [5,6]). Moreover, there are other groups satisfying the conditions, namely $U(n) \subset SO(2n)$ for $n > 2$ and $\mathrm{Spin}(9) \subset SO(16)$. In particular, there is an analogon of Cabrera's result for Spin(9). On the other side, there are interesting G-structures where the group does not satisfy the conditions. The first example $SO(3) \subset SO(5)$ (the irreducible representation) does not admit any invariant differential form. The subgroup $U(2) \subset SO(4)$ admits an invariant form, but the second condition of the Proposition is not satisfied. In these geometries the condition $\mathrm{d}\Gamma = 0$ is an additional requirement on the geometric structure of vectorial type.

Example 2.1. Consider the subgroup $U(2) \subset SO(4)$. There are only two types of U(2)-structures. An almost hermitian manifold (M^4, g, J) is of vectorial type if and only if the almost complex structure is integrable; see [3]. Consequently, starting with an arbitrary complex 4-manifold (M^4, J), *any* hermitian metric g yields a U(2)-structure of vectorial type and the vector field Γ is defined by the formulas

$$d\Omega = -2\Gamma \wedge \Omega, \quad \delta^g \Omega = 2\Gamma \lrcorner \Omega.$$

Solving this algebraic equation, we obtain $2\Gamma = *J(d\Omega)$. In general, this 1-form is not closed. Hermitian manifolds with a closed form Γ are called locally conformal Kähler manifolds. In higher dimensions (i.e. for $G = U(n)$ and $n \geq 3$) all hermitian manifolds of vectorial type are automatically locally conformal Kähler.

Example 2.2. Consider the subgroup $G = SO(n-1) \subset SO(n)$. A G-structure on (M^n, g) is a vector field Ω of length one. The geometric structure is of vectorial type if and only if there exists a vector field Γ such that

$$0 = \nabla^{\text{vec}}_X \Omega = \nabla^g_X \Omega - g(X, \Omega)\Gamma + g(\Omega, \Gamma)X$$

holds. This condition implies that Ω defines a codimension one foliation on M^n,

$$d\Omega = \Omega \wedge \Gamma.$$

Moreover, the second fundamental form of any leave $F^{n-1} \subset M^n$ is given by the formula

$$\mathrm{II}(X) = -g(\Omega, \Gamma) \cdot X, \quad X \in TF^{n-1}.$$

Therefore, the leaves are umbilic. Conversely, let Ω be a 1-form defining an umbilic foliation. Let us define the vector field Γ by the formulas

$$\mathrm{II}(X) = -g(\Omega, \Gamma) \cdot X, \quad d\Gamma = \nabla^g_\Omega \Omega + g(\Omega, \Gamma) \cdot \Omega.$$

Then the induced $SO(n-1)$-structure is of vectorial type and Γ is the corresponding vector field. In consequence, $SO(n-1)$-*structures of vectorial type coincide with umbilic foliations of codimension one*. The vector field Γ satisfies the condition $\Omega \wedge d\Gamma = 0$, but in general it does not have to be closed.

Remark 2.2. An almost contact metric structure $(M^{2k+1}, \xi, \eta, \varphi)$ is never of vectorial type. Indeed, the condition $\eta \wedge (d\eta)^k \neq 0$ contradicts $d\eta = \eta \wedge \Gamma$.

Fix a spin structure of the manifold (M^n, g). The metric connection ∇^{vec} acts on arbitrary spinor fields by

$$\nabla^{\text{vect}}_X \Psi = \nabla^g_X \Psi - \frac{1}{2} \cdot (X \wedge \Gamma) \cdot \Psi.$$

We now consider the case that the group G lifts into the spin group Spin(n) and admits a G-invariant algebraic spinor $\Psi \in \Delta_n$ in the n-dimensional spin representation Δ_n. We normalize the length of the spinor, $\|\Psi\| = 1$. It defines a spinor field on any Riemannian manifold with a G-structure. Moreover, Ψ is parallel with respect to any G-connection. Using this parallel spinor field we can calculate the Riemannian Ricci tensor Ric^g completely. Furthermore, we obtain an algebraic restriction for the 2-form $d\Gamma$.

Theorem 2.1. *Let* $\mathrm{G} \subset \mathrm{SO}(n)$ *be a subgroup lifting into the spin group and suppose that there exists a* G*-invariant spinor* $0 \neq \Psi \in \Delta_n$. *Then the Clifford product* $d\Gamma \cdot \Psi = 0$ *vanishes for any* G*-structure of vectorial type. If the dimension* $n \geq 5$ *is at least five, then* Γ *is closed,* $\mathrm{d}\Gamma = 0$. *The Ricci tensor is given in dimension* $n = 4$ *by*

$$g(\mathrm{Ric}^g(X), Y) = g(\nabla^g_X \Gamma, Y) + g(\nabla^g_Y \Gamma, X) - \delta^g(\Gamma) \cdot g(X, Y) + g(A(X, \Gamma), Y),$$

and in higher dimensions $n \geq 5$ *by*

$$\mathrm{Ric}^g(X) = (n-2)\nabla^g_X \Gamma - \delta^g(\Gamma) \cdot X + A(X, \Gamma).$$

The vector $A(X, \Gamma)$ *is defined by*

$$A(X, \Gamma) := \begin{cases} 0 & \text{if } X \text{ and } \Gamma \text{ are proportional} \\ (n-2)\|\Gamma\|^2 \cdot X & \text{if } X \text{ and } \Gamma \text{ are orthogonal.} \end{cases}$$

The scalar curvature Scal^g *is given by the formula*

$$\mathrm{Scal}^g = 2(1-n)\delta^g(\Gamma) + (n-1)(n-2)\|\Gamma\|^2.$$

Proof. The spinor field Ψ is parallel with respect to the connection ∇^{vec}, i.e.

$$\nabla^g_X \Psi = \frac{1}{2} \cdot (X \wedge \Gamma) \cdot \Psi.$$

We compute the square of the Dirac operator as well as the spinorial Laplacian on Ψ,

$$(D^g)^2 \Psi = \frac{1-n}{2}(\delta^g(\Gamma) + \mathrm{d}\Gamma) \cdot \Psi + \frac{(n-1)^2}{4}\|\Gamma\|^2 \cdot \Psi,$$
$$\Delta \Psi = -\frac{1}{2} \cdot \mathrm{d}\Gamma \cdot \Psi + \frac{n-1}{2}\|\Gamma\|^2 \cdot \Psi.$$

The Schrödinger–Lichnerowicz formula $(D^g)^2 = \Delta + \mathrm{Scal}^g/4$ yields the equation

$$2(1-n)\delta^g(\Gamma) \cdot \Psi + (n-1)(n-2)\|\Gamma\|^2 \cdot \Psi + 2(2-n)\mathrm{d}\Gamma \cdot \Psi = \mathrm{Scal}^g \cdot \Psi.$$

Then we conclude that $\mathrm{d}\Gamma \cdot \Psi = 0$ and

$$2(1-n)\delta^g(\Gamma) + (n-1)(n-2)\|\Gamma\|^2 = \mathrm{Scal}^g.$$

The differential equation for the spinor Ψ allows us to compute the action of the curvature $\mathcal{R}^g(X, Y) \cdot \Psi = \nabla^g_X \nabla^g_Y \Psi - \nabla^g_Y \nabla^g_X \Psi - \nabla^g_{[X,Y]} \Psi$ on the spinor. Then we use the well-known formula (see [11])

$$\mathrm{Ric}^g(X) \cdot \Psi = -2\sum_{i=1}^{n} e_i \cdot \mathcal{R}^g(X, e_i) \cdot \Psi,$$

and after a straightforward algebraic calculation in the Clifford algebra we obtain

$$\mathrm{Ric}^g(X) \cdot \Psi = A(X, \Gamma) \cdot \Psi + (n-3)(\nabla^g_X \Gamma) \cdot \Psi - \delta^g(\Gamma) X \cdot \Psi + \sum_{i=1}^{n} g(X, \nabla^g_{e_i} \Gamma) \cdot e_i \cdot \Psi.$$

Consider the inner product of the latter equation with the spinor $Y \cdot \Psi$. Then we obtain

$$g(\mathrm{Ric}^g(X), Y) = g(A(X, \Gamma), Y) + (n-3)g(\nabla^g_X \Gamma, Y) - \delta^g(\Gamma)g(X, Y) + g(X, \nabla^g_Y \Gamma).$$

Since the Riemannian Ricci tensor Ric^g is symmetric, the antisymmetric part

$$\frac{n-4}{2}\mathrm{d}\Gamma(X,Y)$$

of the right side has to vanish. The formula for the Ricci tensor follows immediately. □

Remark 2.3. The conditions of the latter theorem are satisfied for the groups $\mathrm{G}_2 \subset \mathrm{SO}(7)$ and $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. The subgroups $\mathrm{U}(n) \subset \mathrm{SO}(2n)$ or $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ do *not* satisfy the conditions; there are no invariant spinors.

Remark 2.4. In dimension $n = 4$, the condition $\mathrm{d}\Gamma \cdot \Psi = 0$ defines a three-dimensional subspace $V^3(\Psi) \subset \Lambda^2(\mathbb{R}^4)$ of 2-forms depending on the spinor Ψ. It is the isotropy Lie algebra of the spinor Ψ.

For the special vector $X = \Gamma$, the formula for the Ricci tensor simplifies:

$$\mathrm{Ric}^g(\Gamma) = (n-2)\nabla^g_\Gamma \Gamma - \delta^g(\Gamma)\cdot\Gamma.$$

We multiply the latter equation by the vector field Γ. In this way we obtain the product $g(\mathrm{Ric}^g(\Gamma),\Gamma)$ of the two vectors.

Corollary 2.1. *Suppose that the subgroup* $\mathrm{G} \subset \mathrm{SO}(n)$ *lifts into the spin group and admits an invariant spinor* $0 \neq \Psi \in \Delta_n$. *Then, for any* G*-structure of vectorial type, we have*

$$g(\mathrm{Ric}^g(\Gamma),\Gamma) = \frac{(n-2)}{2}\cdot\Gamma(\|\Gamma\|^2) - \delta^g(\Gamma)\cdot\|\Gamma\|^2.$$

If the manifold M^n *is compact, then*

$$\int_{M^n} g(\mathrm{Ric}^g(\Gamma),\Gamma) = \frac{(n-4)}{2}\cdot\int_{M^n}\delta^g(\Gamma)\cdot\|\Gamma\|^2.$$

The next proposition states that – up to a conformal change of the metric – compact G-structures of vectorial type admitting a parallel spinor are locally conformal to products of Einstein spaces with $\mathbb{R}$. The compactness is an essential assumption here. In [9] the authors constructed non-compact, seven-dimensional solvmanifolds equipped with a G_2-structure of vectorial type that are not Riemannian products of $\mathbb{R}$ with an Einstein space of positive curvature.

Proposition 2.2. *Let* $\mathrm{G} \subset \mathrm{SO}(n)$ *be a subgroup that can be lifted into the spin group and suppose that there exists a spinor* G*-invariant* $0 \neq \Psi \in \Delta_n$. *Consider a* G*-structure of vectorial type on a compact manifold and suppose that* $\delta^g(\Gamma) = 0$ *holds. In dimension* $n = 4$ *we assume moreover that* Γ *is a closed form,* $\mathrm{d}\Gamma = 0$. *Then we have*

(1) $\nabla^g\Gamma = 0$.
(2) $\mathrm{Ric}^g(\Gamma) = 0$.
(3) *If* X *is orthogonal to* Γ, *then* $\mathrm{Ric}^g(X) = (n-1)\cdot\|\Gamma\|^2\cdot X$.
(4) *The scalar curvature is positive:*

$$\mathrm{Scal}^g = (n-1)(n-2)\|\Gamma\|^2 > 0.$$

(5) *The Lie derivative of any* G*-invariant differential form* $\Sigma^k \in \Lambda^k(\mathbb{R}^n)$ *vanishes:*

$$\mathcal{L}_\Gamma\Sigma^k = \nabla^g_\Gamma\Sigma^k = 0.$$

(6) *The universal covering* $\tilde{M}^n = Y^{n-1} \times \mathbb{R}^1$ *splits into* $\mathbb{R}$ *and an Einstein manifold* Y^{n-1} *with positive scalar curvature admitting a real Riemannian Killing spinor.*

Proof. The 1-form Γ is by assumption harmonic and the Bochner formula for 1-forms yields

$$0 = \int_{M^n} \|\nabla^g \Gamma\|^2 + \frac{1}{3}\int_{M^n} g(\mathrm{Ric}^g(\Gamma), \Gamma) = \int_{M^n} \|\nabla^g \Gamma\|^2.$$

Consequently, Γ is parallel with respect to the Levi-Civita connection. Moreover, the restriction of the spinor field Ψ to the submanifold Y^{n-1} defines a spinor field such that

$$\nabla_X^{Y^{n-1}} \Psi = \frac{1}{2} \cdot X \cdot \Gamma \cdot \Psi, \quad \nabla_X^{Y^{n-1}} \Gamma \cdot \Psi = \frac{1}{2}\|\Gamma\|^2 \cdot X \cdot \Psi$$

holds for any vector $X \in T(Y^{n-1})$. The spinor field $\Psi^* := \|\Gamma\| \cdot \Psi + \Gamma \cdot \Psi$ is a Killing spinor on Y^{n-1}. □

Remark 2.5. Let us discuss the latter proposition from the point of view of Weyl geometry. Theorem 2.1 means that any G-structure with a fixed spinor on a compact manifold induces a Weyl–Einstein geometry with a closed form Γ ($n \geq 5$). Indeed, after a conformal change of the metric the condition $\delta^g(\Gamma) = 0$ is satisfied. In this sense, Proposition 2.2 is a reformulation of Theorem 3 in [14]. For dimensions $n = 7$ and $n = 8$, this splitting has been discussed in [15].

3. Geometric structures of vectorial type admitting a characteristic connection

Consider a geometric structure $\mathcal{R} \subset \mathcal{F}(M^n)$ of vectorial type and suppose that it admits a characteristic connection. Then the intrinsic torsion $\Gamma \in \mathbb{R}^n \otimes \mathfrak{m}$ is given by a vector $\Gamma \in \mathbb{R}^n$:

$$\Gamma(X) = \mathrm{pr}_{\mathfrak{m}}(X \wedge \Gamma).$$

On the other hand, there exists a 3-form T^c such that

$$2 \cdot \Gamma(X) = -\Theta(\mathrm{T}^c)(X) = -\mathrm{pr}_{\mathfrak{m}}(X \lrcorner \mathrm{T}^c)$$

holds. Consequently, the vector field Γ and the characteristic torsion T^c are related by the condition

$$2 \cdot (X \wedge \Gamma) + X \lrcorner \mathrm{T}^c \in \mathfrak{g}$$

for all vectors X. In this case, we have two connections ∇^{vec} and ∇^c preserving the G-structure. The map $\mathbb{R}^n \subset \mathbb{R}^n \otimes \mathfrak{m}$ is injective for any subgroup $\mathrm{G} \neq \mathrm{SO}(n)$. If, moreover, the map $\Theta : \Lambda^3(\mathbb{R}^n) \to \mathbb{R}^n \otimes \mathfrak{m}$ is injective too, then the characteristic torsion T^c is uniquely defined by the vector field Γ. Structures with this property and with a non-trivial Γ cannot occur for all geometric structures. Indeed, the G-representation $\mathbb{R}^n$ has to be contained in the G-representation $\Lambda^3(\mathbb{R}^n)$. For example, for the subgroups $\mathrm{G} = \mathrm{SO}(3) \subset \mathrm{SO}(5)$, $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ or $\mathrm{G} = \mathrm{F}_4 \subset \mathrm{SO}(26)$ this condition is not satisfied (see [12]). In dimensions $n = 7, 8$ any G_2- or $\mathrm{Spin}(7)$-structure of vectorial type admits a characteristic connection (see [13,12]).

If the group G preserves a spinor, the corresponding spinor field Ψ on the manifold is parallel with respect to the connections ∇^{vec} and ∇^c. A similar computation to in the proof of Theorem 2.1 yields the following formulas linking Γ, T^c and Ψ.

Theorem 3.1. *Let* $\mathrm{G} \subset \mathrm{SO}(n)$ *be a subgroup lifting into the spin group and suppose that there exists a* G*-invariant spinor* $0 \neq \Psi \in \Delta_n$. *Consider a* G*-structure of vectorial type that admits a* *characteristic connection. Denote by Γ the corresponding vector field and by T^{c} the torsion of the characteristic connection. Then we have*

$$(\Gamma \lrcorner \mathrm{T}^{\mathrm{c}}) \cdot \Psi = 0, \quad \delta(\mathrm{T}^{\mathrm{c}}) \cdot \Psi = 0, \quad \mathrm{T}^{\mathrm{c}} \cdot \Psi = \frac{2}{3}(n-1)\Gamma \cdot \Psi,$$

$$(\mathrm{T}^{\mathrm{c}})^2 \cdot \Psi = \frac{4}{9}(n-1)^2\|\Gamma\|^2 \cdot \Psi,$$

$$\mathrm{dT}^{\mathrm{c}} \cdot \Psi = \frac{1}{3}\left(\|\mathrm{T}^{\mathrm{c}}\|^2 - \frac{4}{9}(n-1)^2\|\Gamma\|^2 - \mathrm{Scal}^{\nabla^{\mathrm{T}^{\mathrm{c}}}}\right) \cdot \Psi,$$

$$2(n-1)\delta^g(\Gamma) = 2\left(\frac{4}{9}(n-1)^2\|\Gamma\|^2 - \|\mathrm{T}^{\mathrm{c}}\|^2\right) - \mathrm{Scal}^{\nabla^{\mathrm{T}^{\mathrm{c}}}}.$$

Example 3.1. Consider a seven-dimensional Riemannian manifold (M^7, g) equipped with a G_2-structure of vectorial type, i.e., with a generic 3-form ω. The differential equations defining the vectorial type of the structure read

$$\mathrm{d}\omega = -3(\Gamma \wedge \omega), \quad \delta(\omega) = 4(\Gamma \lrcorner \omega).$$

The characteristic torsion is given by the formula $\mathrm{T}^{\mathrm{c}} = - * (\Gamma \wedge \omega)$; see [13]. In particular, we have

$$\Gamma \lrcorner \mathrm{T}^{\mathrm{c}} = 0, \quad \delta(\mathrm{T}^{\mathrm{c}}) = 0, \quad \|\mathrm{T}^{\mathrm{c}}\|^2 = 4\|\Gamma\|^2, \quad 12\delta(\Gamma) = 6\|\mathrm{T}^{\mathrm{c}}\|^2 - \mathrm{Scal}^{\nabla^{\mathrm{T}^{\mathrm{c}}}}.$$

4. Generalized Hopf structures

The condition $\nabla^{\mathrm{vec}}\Gamma = 0$ or $\nabla^{\mathrm{vec}}\mathrm{T}^{\mathrm{c}} = 0$ is very restrictive. Indeed, it implies that

$$\delta^g(\Gamma) = (n-1) \cdot \|\Gamma\|^2.$$

Integrating the latter equation over a compact manifold, we obtain $\Gamma \equiv 0$. The conditions $\nabla^{\mathrm{c}}\Gamma = 0$ or $\nabla^{\mathrm{c}}\mathrm{T}^{\mathrm{c}} = 0$ are more interesting (see [3]).

Proposition 4.1. *Suppose that $\Theta : \Lambda^3(\mathbb{R}^n) \to \mathbb{R}^n \otimes \mathfrak{m}$ is injective and let $\mathcal{R}$ be a* G*-structure of vectorial type admitting a characteristic connection. If $\nabla^{\mathrm{c}}\Gamma = 0$, then*

$$\delta^g(\Gamma) = 0, \quad \delta^g(\mathrm{T}^{\mathrm{c}}) = 0, \quad \mathrm{d}\Gamma = \Gamma \lrcorner \mathrm{T}^{\mathrm{c}}, \quad 2 \cdot \nabla^g\Gamma = \mathrm{d}\Gamma.$$

In particular, Γ is a Killing vector field.

Proof. The formulas follow directly from the assumption

$$0 = \nabla^{\mathrm{c}}_X\Gamma = \nabla^g_X\Gamma + \frac{1}{2}\mathrm{T}^{\mathrm{c}}(X, \Gamma, -). \quad \square$$

In complex geometry, a hermitian manifold of vectorial type such that its characteristic torsion T^{c} is ∇^{c}-parallel is called a *generalized Hopf manifold*. These $\mathcal{W}_4$-manifolds have been studied by Vaisman; see [16]. Let us revisit this geometry in more detail.

Example 4.1. Consider the subgroup $\mathrm{U}(2) \subset \mathrm{SO}(4)$. There are only two types of $\mathrm{U}(2)$-structures. Moreover, a $\mathrm{U}(2)$-structure is of vectorial type if and only if it admits a characteristic connection. The link between the 3-form $\mathrm{T^c}$ and the vector field Γ is $\Gamma = *\mathrm{T^c}$ (see [3]). Consequently, we obtain

$$\nabla^{\mathrm{c}}_X \Gamma = \nabla^g_X \Gamma + \frac{1}{2}\mathrm{T^c}(X, \Gamma, -) = \nabla^g_X \Gamma.$$

The condition $\nabla^{\mathrm{c}}_X \Gamma = 0$ is equivalent to $\nabla^g_X \Gamma = 0$. These are generalized Hopf surfaces. They are locally conformal Kähler manifolds ($\mathrm{d}\Gamma = 0$) with a non-parallel vector field ($\nabla^g \Gamma \neq 0$; see [4]). There are also U(2)-structures of vectorial type with a non-closed form Γ (see Example 2.1).

Example 4.2. Consider a hermitian manifold (M^6, g, J) and denote by Ω its Kähler form. The vector field Γ (the vector part of the intrinsic torsion) is defined by

$$\delta^g(\Omega) = 4 \cdot \mathrm{J}(\Gamma) = 4 \cdot (\Gamma \lrcorner \Omega), \quad \mathrm{d}\Omega = -2 \cdot (\Gamma \wedge \Omega).$$

Suppose that M^6 is of vectorial type and that $\nabla^{\mathrm{c}} \Gamma = 0$ holds. Then its characteristic connection as well as the differential are given by the formulas (see [3])

$$\mathrm{T^c} = 2 \cdot (\mathrm{J}(\Gamma) \wedge \Omega), \quad \mathrm{d}\Gamma = 0 = \Gamma \lrcorner \mathrm{T^c}, \quad \nabla^g \Gamma = 0.$$

In particular, Γ is parallel with respect to the Levi-Civita connection and $\mathrm{J}(\Gamma)$ is a Killing vector field.

Definition 4.1. A G-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ of vectorial type and admitting a characteristic connection is called a *generalized Hopf structure* if $\nabla^{\mathrm{c}} \Gamma = 0$ holds.

The vector field Γ of a Hopf G-structure is a Killing vector field. Γ is ∇^g-parallel if and only if $\mathrm{d}\Gamma = 0$ holds. Proposition 2.1 and Theorem 2.1 contain sufficient conditions that the vector field of any Hopf G-structure is ∇^g-parallel. This situation occurs for the standard geometries of the groups $\mathrm{G} = \mathrm{G}_2$, Spin(7) and for $\mathrm{U}(n)$, $n \geq 3$. However, there are subgroups $\mathrm{G} \subset \mathrm{SO}(n)$ and Hopf G-structures ($\nabla^{\mathrm{c}} \Gamma = 0$) with a non-$\nabla^g$-parallel vector field.

Acknowledgements

This work was supported by the SFB 647 “Raum, Zeit, Materie” and the SPP 1154 “Globale Differentialgeometrie” of the DFG as well as the Volkswagen Foundation.

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